Paul Norbury
24.08.2017

$\boxed{\text{Moduli Space of Higgs bundles}}$

— lecture by Paul Norbury

I) Brief comments on spectral curves

II) Higgs bundles, moduli space

Remark: Higgs pair: (P, φ) P: principal G-bundle.
$\varphi \in H^0(\mathfrak{g}_P \otimes K_\Sigma)$
$P \downarrow \Sigma$ alg. curve.

K_Σ: canonical bundle on Σ. It could be an arbitrary line bundle L in general.

III) Hitchin fibration, spectral curve, Hitchin fibre.

I) Spectral curves:

Given a family of $N \times N$ matrices $A(s)$, $s \in \mathbb{C}^{k+1}$.

$$\det(\lambda I - A(s)) = 0 \leftsquigarrow N \text{ fold cover of } \mathbb{C}^{k+1}.$$

Write: $\mathbb{C}^{k+1} = \mathbb{C} \times \mathbb{C}^k$. this is a family of curves parametrised by $\mathbb{C}^k$.

Example: Take $k=0$, $\det(\lambda I - A) = 0$ is an N-fold cover of $\mathbb{C}$.

Hitchin: The spectral curves live somewhere in a complex surface.

Example: (aside)

Magnetic monopoles in $\mathbb{R}^3$: pick out special straight lines in $\mathbb{R}^3$

The space of oriented lines in $\mathbb{R}^3 \cong T\mathbb{P}^1 \supset C \rightsquigarrow$ these special straight lines.
later: spectral curve.

Rmk. Eigenvectors of $A \overset{\text{gives}}{\rightsquigarrow}$ line bundles over C.

Higgs bundles:

Begin with a curve Σ.

Paul Norbury
24.08.2017

Def: A Higgs bundle is a pair (P, φ), where

P is a principal G-bundle, & $\varphi \in H^0(\mathfrak{g}_P \otimes \underbrace{K_\Sigma}_{\text{Canonical line bundle on } \Sigma})$,

(We take $G = GL(N, \mathbb{C})$)

& $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$ is the adjoint bundle.

We work with $G = GL(N, \mathbb{C})$.

Consider a vector bundle E, $\quad (E, \varphi) \leftarrow \varphi \in H^0(\mathrm{End}(E) \otimes K_\Sigma)$

$$E = P \times_G \mathbb{C}^N \quad \text{rank } N \text{ vector bundle on } \Sigma, \qquad \mathrm{End}(E) = \mathfrak{g}_P.$$

$$\boxed{\begin{array}{l}\textbf{Def} \text{ The moduli space of Higgs bundles, for } \Sigma \text{ curve,} \\ \mathcal{M}_\Sigma(N,d) = \left\{ (E, \varphi) \;\middle|\; \begin{array}{l} \mathrm{rank}(E) = N, \quad c_1(E) = d, \\ \varphi \in H^0(\mathrm{End}\, E \otimes K_\Sigma). \end{array} \right\}\end{array}}$$

Stable Higgs bundles $= \{ (E, \varphi) \text{ stable pairs} \}$

stable means: for any φ-invariant subbundle $F \subseteq E$,

$$\mu(F) < \mu(E) = \frac{c_1(E)}{\mathrm{rk}(E)} = \frac{d}{N}.$$

Rmks.

Moduli space of stable bundles $\xleftrightarrow[\text{[Hard]}]{1:1}$ Topological bundles with flat connections.

Calculate the dim:

Hitchin fibration:

$$\mathcal{M}_\Sigma(N,d) \longrightarrow A = \bigoplus_{m=1}^{N} H^0(\Sigma, K_\Sigma^m)$$

$$(E, \varphi) \longmapsto \det(\eta - \varphi) = \eta^N + a_1 \eta^{N-1} + \cdots + a_N,$$

where: $a_m \in H^0(\Sigma, K_\Sigma^m)$.

Paul Norbury
24.08.2017

3

How to think of η ?

$$\begin{array}{ccc} K_\Sigma & & \pi^* K_\Sigma \\ \pi \downarrow \text{ projection,} & & \downarrow \big) \\ \Sigma & & K_\Sigma \end{array}$$

η: tautological section, $\eta \in H^0(K_\Sigma, \pi^* K_\Sigma)$.

$(a_1, a_2, \dots, a_N)$ equivalent to the information $(\mathrm{Tr}(\varphi), \mathrm{Tr}(\varphi^2), \dots, \mathrm{Tr}(\varphi^N))$.

Hitchin base

$$\dim A = \sum_{m=1}^{N} \dim H^0(\Sigma, K_\Sigma^m)$$

$$= 1 + \sum_{m=1}^{N} [1 - g + m(2g-2)]$$

Comes from $m=1$ case.

(No H^1. By RR formula except when $m=1$)

$$= 1 + (g-1) \sum_{m=1}^{N} (2m-1)$$

$$= 1 + (g-1) N^2, \quad g = \text{genus of } \Sigma. \qquad \square$$

What's dim of the Fiber ?

$\boxed{\text{Spectral curve:}}$ $S = \{\det(\eta - \varphi) = 0\} \subseteq K_\Sigma$.

Strictly: $\det(\eta - \pi^*\varphi) = \eta^N + a_1 \eta^{N-1} + \dots + a_N$, $a_j \in H^0(\pi^* K_\Sigma)$.

To calculate genus $g_S = \text{genus}(S)$.

Use adjunction formula [we could compactify the curve if needed]

Note: K_{K_Σ} is trivial.

$$\boxed{-\underbrace{K_{K_\Sigma} \cdot S}_{=\,0} = S.S + \underbrace{\chi(S)}_{=\,2-2g_S}}$$

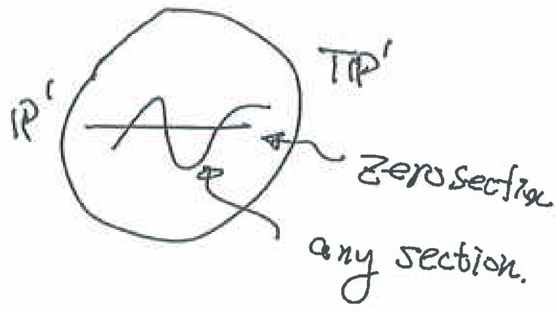
Self intersection. $S.S = N^2(2g-2)$

$$\Rightarrow g_S = 1 + (g-1) N^2.$$

Paul Norbury 24.08.2017 (4)

Example: (about the self intersection.)

$$\begin{array}{c} T\mathbb{P}^1 \\ \downarrow \uparrow \sigma \\ \mathbb{P}^1 \end{array} \qquad c_1(T\mathbb{P}^1) = 2, \quad \sigma\cdot\sigma = 2$$



In our case,

$$\begin{array}{c} K_\Sigma \\ \downarrow \uparrow \sigma \\ \Sigma \end{array} \qquad \sigma\cdot\sigma = 2g-2.$$

We have N-fold cover of Σ. $\Rightarrow S\cdot S = N^2(2g-2)$.

<u>NOW</u>: Fix a point on the Hitchin base. (then the spectral curve S is fixed).

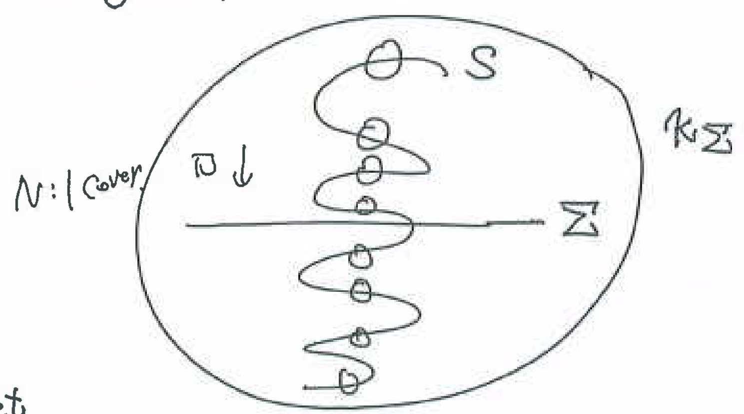
The eigenvector v of $\eta - \varphi$:

$(\eta - \varphi)v = 0$ defines a line bundle $\begin{array}{c} L \subseteq S\times\mathbb{C}^N \\ \downarrow \\ S \end{array}$ on S.

Conversely, for any $\begin{array}{c} L \\ \downarrow \\ S \end{array}$ line bundle. $\begin{array}{c} \pi_* L = E \\ \downarrow \\ \Sigma \end{array}$ rk N vector bundle.

Recall $S \xrightarrow{\pi} \Sigma$ is a deg N map.

<u>Picture</u>:



Let $U \subseteq S$ be an open subset,

Recall: η is the section $\begin{array}{c} \pi^* K_\Sigma \\ \eta \uparrow\downarrow \\ K_\Sigma \supseteq S \end{array}$, then multip. by η is:

$$H^0(U, L) \xrightarrow{\cdot\eta} H^0(U, L\otimes \pi^* K_\Sigma).$$

When we do push forward:

Paul Norbury
24.08.2017

$$\pi_*(L \otimes \pi^* K_\Sigma) \overset{\text{projection formula}}{=\!=\!=} (\pi_* L) \otimes K_\Sigma = E \otimes K_\Sigma$$

$\Rightarrow \pi_* \eta$ gives φ : K_Σ-valued endomorphism. $\Rightarrow$ φ Higgs field!

Summary: Fix a point $a \in A$, let S_a be the spectral curve, then:

$$\boxed{\begin{array}{ccl} \mathrm{Pic}(S_a) & \cong & \mathcal{M}_a \leftarrow \text{Hitchin fiber at } a. \\ \begin{array}{c} L \\ \downarrow \\ S_a \end{array} & \longmapsto & (\pi_* L, \varphi) \text{ as above.} \end{array}}$$

Rmks:

$$c_1(E) = c_1(L) + (g-1)(N^2 - N) \quad [\text{By Grothendieck RR}].$$

Note: $\dim \mathrm{Pic}(S) = g_S = \dim A$.

We have the Lagrangian fibration. $\underset{[\text{Hitchin system}]}{\mathcal{M}(N,d)} = \underset{[\text{moduli of G-bundles}]}{T^* m(N,d)}$

For general Γ, instead of K_Σ:

$$\underset{\substack{\uparrow \\ \text{Principal G-bundle}}}{(P, \varphi)} \qquad \varphi \in H^0(\mathfrak{g}_P \otimes \Gamma), \text{ where } \begin{array}{c} \Gamma \\ \downarrow \\ \Sigma \end{array} \text{ any line bundle.}$$

We make the following changes: $G = GL(N, \mathbb{C})$.

Hitchin fibration: $\mathcal{M}^\Gamma(N,d) \longrightarrow A = \bigoplus_{m=1}^{N} H^0(\Sigma, \Gamma^m)$.

& $\eta \in H^0(\Gamma, \pi^* \Gamma)$.

Spectral curve:

$$\begin{array}{c} S \subseteq \Gamma. \\ \| \\ \{\det(\eta - \varphi)\} \end{array}$$

Let $\deg \Gamma := r + 2g - 2$, then $\dim A = N^2(g-1) + \frac{1}{2} N(N+1) r$.

Paul Norbury ⑥
24.08.2017

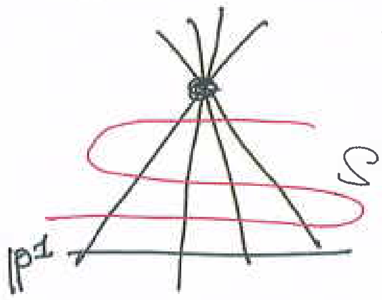
But now: $K_P \neq \mathcal{O}$, in general.

P ↑ complex surface

<u>Example</u>: $\Sigma = \mathbb{P}^1$, $N=3$, $P = \mathcal{O}(1)$.

the total space of P is: $\mathbb{P}^2 \setminus pt$, say $pt = [0,0,1]$.

$$\mathbb{P}^2 \setminus pt \longrightarrow \mathbb{P}^1 \qquad [x,y,t] \mapsto [x,y].$$



Consider $\mathcal{M}(3,d)$.

$$\dim A = \sum_{m=1}^{3} \dim H^0(\mathcal{O}(1)^{\otimes m}) = 2+3+4 = 9.$$

$$K_{\mathcal{O}(1)} = -3\,\mathbb{P}^1 \;(= K_{\mathbb{P}^2})$$

By adjunction, $g_S = \frac{1}{2}(N-1)(N-2)$

$= 1$ (since $N=3$)

Take the spectral curve:

$$S = \{ y^2 z - x^3 - a_2 x z^2 - a_3 z^3 = 0 \} \subseteq \mathbb{P}^2 \setminus [0,0,1].$$

with $a_3 \neq 0$.

Let $\begin{matrix} L \\ \downarrow \\ S \end{matrix}$ be a line bundle, then: $\pi_* L$ is a rank 3 bundle over $\mathbb{P}^1$.

$$S \xrightarrow{\pi} \mathbb{P}^1$$

Fix x, y, then $y^2z - x^3 - a_2 x z^2 - a_3 z^3 = 0$ has 3 solutions: z_1, z_2, z_3.

$$\begin{array}{c} \pi^*\Gamma = \pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^2}(1), \quad \text{and } \eta = z \in H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \\ \downarrow \\ \Gamma = \mathcal{O}_{\mathbb{P}^1}(1) \end{array}$$

$\pi_* L$ is a rk 3 bundle$/\mathbb{P}^1$, fiber $\mathbb{C}^3$.

At (x, y, z) is evaluation of function at z_1, z_2, z_3.

$$\underset{\text{Higgs field}}{\varphi} \in \mathrm{End}(\mathbb{C}^3) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \longleftrightarrow z \text{ acts on } \mathbb{C}^3 \text{ by } \underset{H^0(\mathcal{O}_{\mathbb{P}^2}(1))}{\underset{\cap}{\begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{bmatrix}}}.$$

□

Some quotes from [D. Nadler: The geometric Nature of the fund. lemma]

$\boxed{\text{Ngô:}}$ Laumon's approach to $\underset{\underset{\text{(affine Springer fibers)}}{\simeq}}{\widehat{Fl}_a}$ via compactified Jacobian is a natural piece of Hitchin fibration.

- The Hitchin fibration is the natural generalization of:

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{t}/\!/W = \mathfrak{g}/\!/G = \mathrm{Spec}\, k[\mathfrak{t}]^W.$$

$\mathfrak{t}$ = Cartan of $\mathfrak{g}$.

to the setting of alg. curves

For $\varphi \in \Gamma(\Sigma, \mathfrak{g}_P \otimes \mathcal{L})$, $\underset{\text{Hitchin base}}{A} = \Gamma(\Sigma, \underbrace{\mathcal{L} \times_{\mathbb{G}_m} \mathfrak{t}/\!/W}_{\text{affine bundle}})$ = the space of all possible eigenvalues, since $\mathcal{O}(\mathfrak{t}/\!/W = \mathfrak{g}/\!/W)$ = the symmetric polynomials and coeffs of $\det(\eta - \varphi) = 0$ lie in $k[\mathfrak{t}]^W$.

Paul Norbury
24.08.2017

Let $a \in A$ be generically regular.

Let $c_i \in \Sigma$ be the finitely many points, where a is not regular.

Let $D_i = \operatorname{Spec} \mathcal{O}_i$ be the formal disc around those c_i.

Kostant slice:

$$(\mathfrak{t}/\!/W)_{\mathcal{O}} \xrightarrow{\varepsilon} \mathfrak{g}_{\mathcal{O}}.$$
$$a|_{D_i} \longmapsto \xi_i$$

The WEAVING Theorem

Theorem [Ng5]

There is a canonical map inducing a topological equivalence:

$$\boxed{\prod_{i \in I} \mathrm{Gr}^{\xi_i} / P_{\xi_i} \xrightarrow{\sim} \mathcal{M}^a / \mathcal{P}_a}$$

affine Springer fiber at $\xi_i \in \mathfrak{g}_{\mathcal{O}}$

local Picard gp

Hitchin fiber at a

global Picard gp.

□

Recall the notations in the above theorem.

$\mathcal{K} = k((t))$, $\mathcal{O} = k[[t]]$ associated to a formal disc D.

- $\mathrm{Gr}^{\xi} := \{ g \in G_{\mathcal{K}} \mid \mathrm{Ad}(g)(\xi) \in \mathfrak{g}_{\mathcal{O}} \} / G_{\mathcal{O}}.$

The centralizer $(G_{\mathcal{K}})_{\xi} \curvearrowright \mathrm{Gr}^{\xi}$

$$(G_{\mathcal{K}})_{\xi} = \{ g \in G_{\mathcal{K}} \mid \mathrm{Ad}(g)\xi = \xi \}$$

Paul Norbury
24.08.2017

$(G_K)_\xi$ canonically extends to a smooth commutative group scheme J over $\mathcal{O}$.

Meaning: On regular part $\mathfrak{g}^{reg} \subseteq \mathfrak{g}$, we have the smooth group scheme of centralizers $\begin{array}{c} I \\ \downarrow \\ \mathfrak{g}^{reg} \end{array}$, then take

$$\begin{tikzcd} J := \mathrm{Spec}\,\mathcal{O} \times_{\mathfrak{g}} I \arrow[r] \arrow[d] & I \arrow[d] \\ \mathrm{Spec}\,\mathcal{O} \arrow[r, "\xi"] \arrow[ur, phantom, "\lrcorner"] & \mathfrak{g}^{reg} \end{tikzcd}$$

- The local picard gp is: $P_\xi := J_K / J_{\mathcal{O}}$.
- The Hitchin fiber
  $$\boxed{M^a \cong \overline{\mathrm{Pic}}(S_a) \quad \text{spectral curve.}}$$
- The global picard gp is: $P_a := \mathrm{Pic}(S_a)$.

$\square$

Hitchin Fibration connects to Spectral Curve.